

ON G -MODULAR FUNCTOR

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ABSTRACT. It is a known result that given a \mathcal{C} extended modular functor where \mathcal{C} is a semisimple abelian category, we can find a structure of weakly rigid fusion category on \mathcal{C} . Also if we have a structure of a weakly rigid fusion category on \mathcal{C} then from this we can define a \mathcal{C} extended modular functor. In this paper, we extend this notion of modular functor and fusion category to what we called G equivariant modular functor and G equivariant fusion category where G is a finite group. Then we establish a similar correspondence between G equivariant modular functor and G equivariant fusion category.

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1. INTRODUCTION

This paper is a continuation of [K4] and [TA]. Its goal is to develop a formalism of G -equivariant modular functors, which would provide a suitable algebraic formalism for orbifold models in conformal field theory, much as usual modular functors can be used for describing various structures appearing in usual conformal field theory. We will also establish a relation of this approach to the theory of G -equivariant fusion categories as defined in [K4] (following earlier work of Turaev [T2]). The main result of this paper is that the notion of G -equivariant modular functor (in genus zero) and a structure of a G -equivariant fusion category are essentially equivalent; precise statement is given in the main theorem of this paper.

Our approach only discusses topological setting, in which the main objects are oriented surfaces with boundary (or, in G -equivariant case, G -covers of such curves).

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Complex-analytic analog, which uses the language of flat connections on the moduli spaces of curves, will be discussed in a subsequent paper.

It should be noted that some of the results here are parallel to the results of [T2]. However, unlike Turaev, our approach is not based on 3D TQFT, in which the main technical tool is presentation of 3-manifolds via surgery and using Kirby moves. Instead, we follow the approach suggested (for non- G -equivariant case) by Moore and Seiberg, presenting a surface as a result of gluing of “standard” spheres with holes, and then writing the generators and relations in the groupoid of all such presentations.

2. G -EQUIVARIANT FUSION CATEGORY

Throughout this paper, G is a fixed finite group.

Definition 2.1. A G -equivariant category is an abelian category \mathcal{C} with the following additional structure:

G -grading: Decomposition

$$\mathcal{C} = \bigoplus_{g \in G} \mathcal{C}_g$$

where each \mathcal{C}_g is a full subcategory of \mathcal{C} . We will call objects $V \in \mathcal{C}_g$ “ g twisted”. In particular, objects $V \in \mathcal{C}_1$ will be called “neutral”. In physical literature, the subcategory \mathcal{C}_1 is usually called the “untwisted sector”.

Action of G : For each $g \in G$, we are given a functor $R_g: \mathcal{C} \rightarrow \mathcal{C}$, and for each pair $g, h \in G$, a functorial isomorphism $\alpha_{g,h}: R_g \circ R_h \rightarrow R_{gh}$. These functorial isomorphisms must satisfy the following conditions:

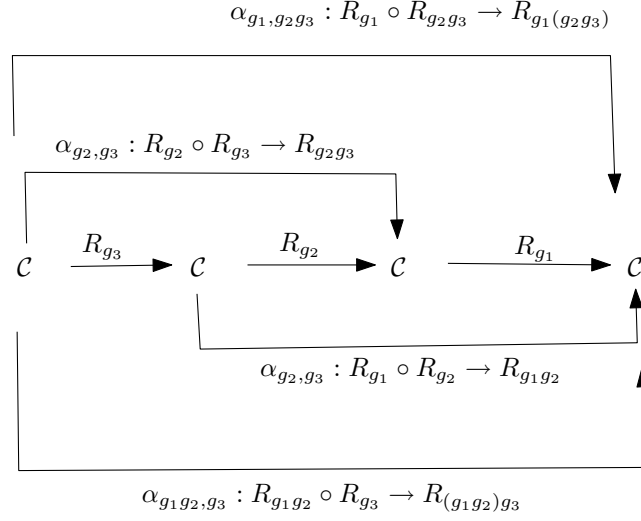
- (1) $R_1 = \text{id}$
- (2) $R_g(\mathcal{C}_h) \subseteq \mathcal{C}_{ghg^{-1}}$. This means the action of G respects the grading.
- (3) $\alpha_{g_1 g_2, g_3} \circ \alpha_{g_1, g_2} = \alpha_{g_1, g_2 g_3} \circ \alpha_{g_2, g_3}$. Here both sides are functorial isomorphism from $R_{g_1} R_{g_2} R_{g_3} \rightarrow R_{g_1 g_2 g_3}$ (see Figure 1). This might be thought of as the associativity of G action.

We will frequently use notation ${}^g V$ for $R_g(V)$.

This definition was introduced by Turaev [T2] under a different name.

Definition 2.2. A G -equivariant fusion category is a semisimple G equivariant abelian category over the base field \mathbb{C} with the following additional structure:

- (1) Structure of a monoidal category such that
 - $\mathbf{1}$ is a simple object
 - for any simple object V_i , $\text{End}_{\mathcal{C}}(V_i) = \mathbb{C}$
 - R_g is a tensor functor
 - For $X \in \mathcal{C}_g$ and $Y \in \mathcal{C}_h$, $X \otimes Y \in \mathcal{C}_{gh}$.
- (2) Structure of rigidity: each object V has a right dual V^* , with evaluation and coevaluation map, $e_V: V^* \otimes V \rightarrow \mathbf{1}$ and $i_V: \mathbf{1} \rightarrow V \otimes V^*$ which satisfy the following rigidity conditions:
 - $(\text{id} \otimes e_V) \circ (i_V \otimes \text{id})$ is the identity map on V
 - $(e_V \otimes \text{id}) \circ (\text{id} \otimes i_V)$ is the identity map on V^*
 - For detailed discussion on this, see e.g. [BK2].
- (3) Functorial isomorphism $\delta_V: V \rightarrow V^{**}$, satisfying the same compatibility conditions as in the absence of G . For detailed discussion see [BK2]. Here, for readers convenience, we list these conditions:

FIGURE 1. Associativity of G action

$$\delta_{V \otimes W} = \delta_V \otimes \delta_W$$

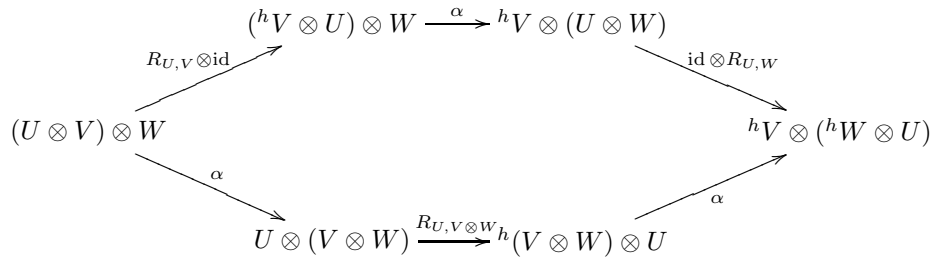
$$\delta_1 = \text{id}$$

$$\delta_{V^*} = (\delta_V^*)^{-1}$$

In addition to the above relations, we also require

$$R_g(\delta_V) = \delta_{R_g(V)}$$

- (4) A collection of functorial isomorphisms, $R_{V,W} : V \otimes W \rightarrow {}^g W \otimes V$ for every $V \in \mathcal{C}_g$ and $W \in \mathcal{C}_h$. This is similar to the braiding in the $G = \{1\}$ case but with the addition of the twist. These functorial isomorphisms must satisfy an analog of two hexagon axioms. The first hexagon axiom is shown in Figure 2; the other one is similar but with R replaced by R^{-1} .

FIGURE 2. Hexagon axiom ($U \in \mathcal{C}_h$)

The definition immediately implies that $1 \in \mathcal{C}_1$ and if $V \in \mathcal{C}_g$ then $V^* \in \mathcal{C}_{g^{-1}}$. Also, since in a rigid monoidal category the unit object and dual is unique up to a unique isomorphism, we have canonical identification

$${}^g 1 = 1$$

$$({}^g V)^* = {}^g(V^*).$$

Remark 2.3. From now on we will refer to the isomorphisms described above: associativity, unit, braiding, rigidity, δ morphism, G -action, and their compositions as canonical morphisms. We will omit these canonical morphisms in the formulas, writing, e.g., $V \otimes U \otimes W$ rather than $(V \otimes U) \otimes W$. Thus all formulas and identities only make sense after the insertion of appropriate canonical morphisms. Pedantic readers may complete all computations by inserting appropriate canonical morphisms.

As in the $G = 1$ case, existence of morphism $\delta : V \rightarrow V^{**}$ is equivalent to existence of the twist θ_V . Hence we have the following lemma.

Lemma 2.4. *Let \mathcal{C} be a G -equivariant fusion category. Then one can define a collection of functorial morphisms $\theta_V : V \rightarrow {}^g V$ for $V \in \mathcal{C}_g$ which satisfy the following conditions (here $V \in \mathcal{C}_g$ and $U \in \mathcal{C}_h$):*

$$\begin{aligned}\theta_1 &= \text{id} \\ \theta_{U \otimes V} &= R_{h,gV,hU} R_{hU,gV} (\theta_U \otimes \theta_V) \\ \theta_{V^*} &= R_{g^{-1}}(\theta_V^*) \\ \theta_{hV} &= R_h(\theta_V).\end{aligned}$$

Conversely, if we have θ satisfying the above condition then we can recover δ from this θ , R , and the monoidal structure.

Proof. The proof is completely parallel to the $G = 1$ case discussed in [BK2, Section 2.2]. Details of G -equivariant case can be found in [K4]. \square

Definition 2.5. Let \mathcal{C} be a G -equivariant category. An object W is called a weak dual of V if

$$\text{Hom}(\mathbf{1}, V \otimes X) = \text{Hom}(W, X).$$

In other words, the functor $\text{Hom}(\mathbf{1}, V \otimes -)$ is represented by the object W . In this case, we will denote W by V^* . Obviously, in the case of a G equivariant fusion category the usual right dual of an object is also a weak dual.

Definition 2.6. A G -equivariant category \mathcal{C} is called a G -equivariant weakly fusion category if it satisfies all the conditions of a G -equivariant fusion category except for the rigidity condition. Instead of the condition that each object has a right dual we require that each object has a weak dual.

Remark 2.7. Of course, a G -equivariant fusion category is also a G -equivariant weakly fusion category but the converse is not true.

3. G -COVERS

The main topological object of our study is the notion of a G -cover of a surface. Detailed description of them is given in the [TA]. For readers convenience, we recall basic definitions here.

Definition 3.1. An *extended surface* is a compact, smooth, oriented, closed surface Σ (not necessarily connected), possibly with boundary and with a choice of a distinguished (marked) point on each of its boundary components.

We denote by $A(\Sigma)$ the set of the boundary components of Σ . So an extended surface will be denoted by $(\Sigma, \{p_a\}_{a \in A(\Sigma)})$ where p_a is the choice of marked point on the a th boundary component.

Definition 3.2. A G cover of $(\Sigma, \{p_a\})$ is a pair $(\pi: \tilde{\Sigma} \longrightarrow \Sigma, \{\tilde{p}_a\})$ where $(\pi: \tilde{\Sigma} \longrightarrow \Sigma)$ is a principal G -cover (possibly not connected) and $\{\tilde{p}_a\}$ are choice of points in the fiber of p_a : $\tilde{p}_a \in \pi^{-1}(p_a)$ for all $a \in A(\Sigma)$.

For brevity, we will usually denote a G -cover just by $\tilde{\Sigma}$, suppressing all other data.

Note that G -covers are not required to be connected but are required to be unbranched.

One can easily define the notion of a morphism between two G -covers; also, since each G -cover comes with a choice of a marked points, we can define, for each boundary component a of Σ , the monodromy $m_a(\tilde{\Sigma}) \in G$ (see, e.g., [TA] for details).

Lemma 3.3. *Let $(\tilde{\Sigma}, \{\tilde{p}_a\})$ be a G -cover, and let $a, b \in A(\Sigma)$ be two boundary components. Let $\varphi: (\partial\Sigma)_a \rightarrow (\partial\Sigma)_b$ be an orientation reversing homeomorphism of boundary circles such that $\varphi(p_a) = p_b$ (it is well-known that such a homeomorphism is unique up to isotopy). Then φ can be lifted to an orientation-reversing homeomorphism of boundary components of the G -cover $\tilde{\varphi}: (\partial\tilde{\Sigma})_a \rightarrow (\partial\tilde{\Sigma})_b$ such that $\tilde{\varphi}(\tilde{p}_a) = \tilde{p}_b$ iff the monodromy satisfies the following condition:*

$$(3.1) \quad m_a m_b = 1.$$

If this condition is satisfied, then one can form a new G -cover by identifying (“gluing”) $(\partial\tilde{\Sigma})_a \rightarrow (\partial\tilde{\Sigma})_b$ using $\tilde{\varphi}$. We will denote this new G -cover by

$$\sqcup_{a,b} \tilde{\Sigma}.$$

As a special case, we can consider the situation where Σ is disconnected: $\Sigma = \Sigma_1 \sqcup \Sigma_2$, and $a \in A(\Sigma_1)$, $b \in (\Sigma_2)$. In this case we will use the notation

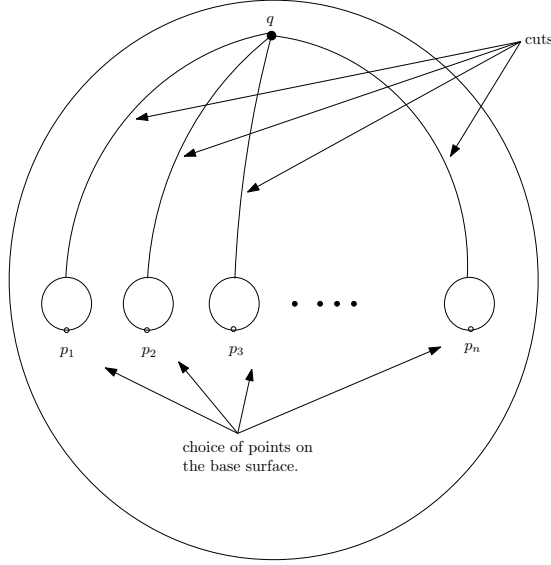
$$\Sigma_1 \sqcup_{a,b} \Sigma_2$$

for the result of identification, or “gluing”.

Definition 3.4. For every $n \geq 0$, we define the standard sphere, S_n , to be the Riemann sphere \mathbb{C} with n disks $|z - k| < \frac{1}{3}$ removed and with the marked points being $k - \frac{i}{3}$, here $k = 1, 2, 3, \dots, n$. Of course, we could replace these n disks with any other n non-overlapping disks with centers on the real line and with marked points in the lower half plane. Any two such spheres are homeomorphic and the homeomorphism can be chosen canonically up to homotopy. Note that the set of boundary components of the standard sphere is naturally indexed by numbers $1, 2, \dots, n$.

We will now define the notion of standard blocks, or some “distinguished” G -covers of the standard sphere. Let us start with the standard sphere with n holes, S_n and $2n$ elements $\{g_1, \dots, g_n\}$ and $\{h_1, \dots, h_n\}$ in G such that $g_1 \dots g_n = 1$. Make the cuts on S_n as in Figure 3. Here the point $q \in S_n$ in Figure 3 is the point at $i\infty$. In fact q can be chosen to be any point on the upper hemisphere as long as it does not belong to the boundary circles. Then one can easily see that $S_n \setminus \text{cuts}$ is simply connected.

Consider the trivial G -cover $(S_n \setminus \text{cuts}) \times G \rightarrow S_n \setminus \text{cuts}$. Now, define a G -cover of S_n by gluing along i -th cut, identifying point $(z, x) \in (S_n \setminus \text{cuts}) \times G$ on the left hand side of the i -th cut with the point (z, xg_i) on the right-hand side. One easily sees that this agrees with the action of G on the fibers (recall that G acts by

FIGURE 3. Cuts on S_n

left multiplication); condition $g_1 \dots g_n = 1$ ensures that this gluing defines a cover which is unbranched at point q . Finally, define a marked point \tilde{p}_i on i -th boundary circle to be $\tilde{p}_i = (p_i, h_i)$. This defines a G -cover of S_n .

Definition 3.5. The G cover of S_n constructed above will be called the standard block and will be denoted by $S_n(g_1, g_2, \dots, g_n; h_1, h_2, \dots, h_n)$. Note that it is only defined if $g_1 g_2 \dots g_n = 1$.

Lemma 3.6. Let $S_n(g_1, \dots, g_n; h_1, \dots, h_n)$ and $S_n(g'_1, \dots, g'_n; h'_1, \dots, h'_n)$ be two standard blocks. Then the identity isomorphism $S_n \rightarrow S_n$ can be lifted to an isomorphism $S_n(\mathbf{g}, \mathbf{h}) \rightarrow S_n(\mathbf{g}', \mathbf{h}')$ iff there exists $x \in G$ so that $xg_i x^{-1} = g'_i$ and $h_i x^{-1} = h'_i$ for $i = 1 \dots n$; in this case, the isomorphism is unique. We denote the isomorphism $S_n(\mathbf{g}, \mathbf{h}) \rightarrow S_n(\mathbf{g}', \mathbf{h}')$ by ϕ_x .

Proof. See the paper [TA]. □

Lemma 3.7. For the standard block $S_n(g_1, \dots, g_n; h_1, \dots, h_n)$, the monodromy $m_i \in G$ around the i -th boundary circle is given by

$$m_i = h_i g_i^{-1} h_i^{-1}.$$

Proof. See the paper [TA]. □

Corollary 3.8. i -th boundary circle of $S_n(g_1, \dots, g_n; h_1, \dots, h_n)$ can be glued to the j -th boundary circle of $S_m(u_1, \dots, u_m; v_1, \dots, v_m)$ iff $h_i g_i^{-1} h_i^{-1} = [v_j u_j^{-1} v_j^{-1}]^{-1}$.

Definition 3.9. Let $\tilde{\Sigma} \rightarrow \Sigma$ be a G cover. A parameterization of $\tilde{\Sigma}$ is an isomorphism of this G -cover with one or gluing of several standard blocks:

$$f: \tilde{\Sigma} \xrightarrow{\sim} S_{n_1}(\mathbf{g}^1, \mathbf{h}^1) \sqcup_{i_1, j_1} S_{n_2}(\mathbf{g}^2, \mathbf{h}^2) \sqcup_{i_2, j_2} \dots S_{n_k}(\mathbf{g}^k, \mathbf{h}^k)$$

It is easy to see that parameterization can be equivalently described by the following data:

- (1) A finite set $C = \{c_1, \dots\}$ of closed non-intersecting curves (“cuts”) $c_i \in \Sigma$.
- (2) A choice of marked points $p_c \in c$, one point for each cut c , and a choice of lifting $\tilde{p}_c \in \pi^{-1}(p_c)$.
- (3) For every connected component Σ_k of $\Sigma \setminus \{c_i\}$, an isomorphism of G -covers $f_k: \tilde{\Sigma}_k \rightarrow S_{n_k}(\mathbf{g}^k, \mathbf{h}^k)$.

Finally, note that if Σ is an extended surface with the set of boundary components $A = A(\Sigma)$, then we have a natural action of the group G^A on the category of G -covers of Σ by changing the marked points \tilde{p}_a : if $\mathbf{x} = \{x_a\}_{a \in A} \in G^A$, then we define

$$(3.2) \quad \mathbf{x}(\tilde{\Sigma}, \{\tilde{p}_a\}) = (\tilde{\Sigma}, \{\tilde{p}'_a\}), \quad \tilde{p}'_a = x_a \tilde{p}_a.$$

4. G -EQUIVARIANT MODULAR FUNCTOR

In this section we introduce the G -equivariant analog of the notion of modular functor; we will call such an analog a *G -equivariant modular functor*, or simply G -MF.

Let \mathcal{C} be a semisimple abelian G equivariant category; we assume that the set I of equivalence classes of simple objects is finite.

Definition 4.1. Let \mathcal{C} be a G -equivariant category. An object $\mathcal{R} \in \mathcal{C} \boxtimes \mathcal{C}$ is called symmetric and G -invariant if

- (1) $\mathcal{R} \in \bigoplus_h \mathcal{C}_h \boxtimes \mathcal{C}_{h^{-1}}$
- (2) \mathcal{R} is symmetric, i.e. we have an isomorphism $\sigma: \mathcal{R} \xrightarrow{\sim} R^{op}$ as in [BK2, Section 2.4].
- (3) For every G we have an isomorphism $\mathcal{R} \simeq (R_g \boxtimes R_g)(\mathcal{R})$; these isomorphisms should be compatible with each other and with the symmetry isomorphism σ .

As we will show later, a typical example of such an object is when \mathcal{C} is a G -fusion category and $\mathcal{R} = \bigoplus_i V_i \boxtimes V_i^*$, where V_i are simple objects.

We will frequently use the following standard convention: if $\mathcal{R} = \bigoplus_i \mathcal{R}_i^1 \boxtimes \mathcal{R}_i^2$, then we will drop the index i and summation from our formulas, writing, for example

$$(4.1) \quad \text{Hom}(A, \mathcal{R}^1) \otimes \text{Hom}(B, \mathcal{R}^2)$$

for $\bigoplus_i \text{Hom}(A, \mathcal{R}_i^1) \otimes \text{Hom}(B, \mathcal{R}_i^2)$.

The following definition is the main definition of this paper; it generalizes the well-known definition of the modular functor to G -equivariant case.

Definition 4.2. Let \mathcal{C}, \mathcal{R} be as above. A \mathcal{C} -extended G -equivariant modular functor (G -MF for short) is the following collection of data:

- (1) To every G -cover $(\tilde{\Sigma}, \{\tilde{p}_a\})$ is assigned a polylinear functor

$$\tau(\tilde{\Sigma}): \boxtimes_{a \in A(\Sigma)} \mathcal{C}_{m_a^{-1}(\tilde{\Sigma})} \rightarrow \mathcal{V}ec.$$

(Here m_a is monodromy around a -th boundary component of the G -cover).

In other words, for every choice of objects $W_a \in \mathcal{C}_{m_a^{-1}}(\tilde{\Sigma})$ attached to every boundary component of Σ is assigned a finite-dimensional vector space $\tau(\tilde{\Sigma}; \{W_a\})$, and this assignment is functorial in W_a .

- (2) To every morphism of G -covers $f: \tilde{\Sigma} \xrightarrow{\sim} \tilde{\Sigma}'$ is assigned a functorial isomorphism $f_*: \tau(\tilde{\Sigma}) \xrightarrow{\sim} \tau(\tilde{\Sigma}')$, which depends only on the isotopy class of f .
- (3) Functorial isomorphisms $\tau(\emptyset) \xrightarrow{\sim} k$, $\tau(N_1 \sqcup N_2) \xrightarrow{\sim} \tau(N_1) \otimes \tau(N_2)$.
- (4) **Gluing isomorphism:** Let $(\tilde{\Sigma}, \{p_a\})$ be a G -cover and $\alpha, \beta \in A(\Sigma)$, $\alpha \neq \beta$ be two boundary components such that condition (3.1) holds. Let $\sqcup_{\alpha, \beta}(\tilde{\Sigma})$ be the surface obtained by gluing components α, β of $\tilde{\Sigma}$ as in Lemma 3.3. Then we have a functorial isomorphism

$$(4.2) \quad G_{\alpha, \beta}: \tau(\tilde{\Sigma}; \{W_a\}, \mathcal{R}_{\alpha, \beta}) \xrightarrow{\sim} \tau(\sqcup_{\alpha, \beta} \tilde{\Sigma}; \{W_a\}),$$

where $\mathcal{R}_{\alpha, \beta}$ means that we assign the symmetric object $\mathcal{R} \in \mathcal{C}^{\boxtimes 2}$ to boundary components α, β .

- (5) For any G -cover $(\tilde{\Sigma}, \tilde{p}_a)$ and any $\mathbf{x} = \{x_a\}_{a \in A(\Sigma)} \in G^{A(\Sigma)}$, we have functorial isomorphisms

$$(4.3) \quad T_{\mathbf{x}}: \tau(\tilde{\Sigma}, \{\tilde{p}_a\}, \{W_a\}) \simeq \tau(\tilde{\Sigma}, \{x_a \tilde{p}_a\}, \{x_a W_a\})$$

The above data have to satisfy the following axioms:

Multiplicativity: $(fg)_* = f_* g_*$, $\text{id}_* = \text{id}$.

Functoriality: All isomorphisms in parts 3, 4, 5 above are functorial in $\tilde{\Sigma}$.

Compatibility: All isomorphisms in parts 3, 4, 5 above are compatible with each other.

Symmetry of gluing: After the identification $\mathcal{R} \simeq \mathcal{R}^{op}$, we have $G_{\alpha, \beta} = G_{\beta, \alpha}$.

Normalization: $\tau(S^2 \times G) = k$.

Explicit statements of all functoriality and compatibility axioms are similar to the ones in $G = \{1\}$ case which can be found in [BK2]. The only new compatibility relations are those involving $T_{\mathbf{x}}$. For the most part, they are quite obvious; the only one which is not immediately obvious is the one involving compatibility of $T_{\mathbf{x}}$ and gluing, which is given below.

Let $\tilde{\Sigma}$ be a G -cover and $\alpha, \beta \in A(\Sigma_1)$, $\alpha \neq \beta$. Assume that condition (3.1) is satisfied so that we can glue α and β boundary components. Let $W_a, a \in A' = A(\Sigma) \setminus \{\alpha, \beta\}$, be a collection of objects from \mathcal{C} and let $\mathbf{x} \in G^A$ be such that $x_\alpha = x_\beta$. Then the following diagram commutes:

$$\begin{array}{ccc} \tau[\tilde{\Sigma}, \{\tilde{p}_a\}; \{W_a\}_{a \in A'}, \mathcal{R}^1, \mathcal{R}^2] & \xrightarrow{G_{\alpha, \beta}} & \tau[\sqcup_{\alpha, \beta} \tilde{\Sigma}, \{\tilde{p}_a\}; \{W_a\}] \\ T_{\mathbf{x}} \downarrow & & \downarrow T_{\mathbf{x}'} \\ \tau[\tilde{\Sigma}, \{x_a \tilde{p}_a\}; \{x_a W_a\}, {}^x \mathcal{R}^1, {}^x \mathcal{R}^2] & \xrightarrow{G_{\alpha, \beta}} & \tau[\sqcup_{\alpha, \beta} \tilde{\Sigma}, \{x_a \tilde{p}_a\}; \{x_a W_a\}] \end{array}$$

Here we assigned \mathcal{R}^1 and \mathcal{R}^2 to α and β boundary components respectively; the bottom arrow also uses isomorphism ${}^x \mathcal{R}^1 \boxtimes {}^x \mathcal{R}^2 \simeq \mathcal{R}^1 \boxtimes \mathcal{R}^2$ (see Definition 4.1).

Also note that condition $x_\alpha = x_\beta$ ensures that if we can glue the α and β boundary components of $(\tilde{\Sigma}, \{\tilde{p}_a\})$ then we can also glue the α and β boundary components of $(\tilde{\Sigma}, \{x_a \tilde{p}_a\})$.

We leave all the other compatibility conditions, which are rather obvious, to the imagination of the readers.

Definition 4.3. A G -MF is called non-degenerate if for every non-zero object $X \in \mathcal{C}$, there exists a G -cover $(\tilde{\Sigma}, \{\tilde{p}_a\})$ and a collection of objects $\{W_i\}$ such that $\tau(\tilde{\Sigma}, \{\tilde{p}_a\}; \{X, W_1, \dots\}) \neq 0$.

As in $G = \{1\}$ case, it is sometimes convenient to consider more restricted version of modular functor, in which we only allow genus 0 surfaces.

Definition 4.4. A genus zero \mathcal{C} extended G -equivariant modular functor (genus 0 G -MF for short) consists of the same data as defined in Definition 4.2 except that τ is only defined for surfaces all connected components of which have genus zero, and the gluing is only defined if α, β are in different connected components of Σ .

In this paper, we will only consider genus zero \mathcal{C} extended G -equivariant modular functor; case of positive genus will be addressed in forthcoming papers.

5. STATEMENT OF THE MAIN THEOREM

The following is the main theorem of this paper. The rest of the paper is devoted to prove this theorem.

Theorem. *Let \mathcal{C} be a semisimple G -equivariant abelian category \mathcal{C} and let*

$$\{V_i \mid i \in I\}$$

be a set of representative of isomorphism classes of simple objects. Assume that $|I| < \infty$.

- (1) *If we have a non-degenerate \mathcal{C} -extended genus zero G -MF, then we can construct a structure of G -equivariant weakly fusion category on \mathcal{C} .*
- (2) *Conversely, if we have a structure of G equivariant fusion (or weakly fusion) category on \mathcal{C} then we can define a non-degenerate \mathcal{C} extended genus zero G -MF.*
- (3) *The above two constructions are inverse of each other.*

The above correspondence between modular functor and fusion category satisfies the following properties:

- (1) *The G -invariant symmetric object \mathcal{R} used in the definition of the MF is given by $\mathcal{R} = \bigoplus V_i \boxtimes V_i^*$.*
- (2) *Let $S_n(\mathbf{g}, \mathbf{h})$ be the standard block as defined in Definition 3.5. Then*

$$\tau[S_n(\mathbf{g}, \mathbf{h}); W_1, \dots, W_n] = \text{Hom}_{\mathcal{C}}(\mathbf{1}, {}^{h_1^{-1}}W_1 \otimes \dots \otimes {}^{h_n^{-1}}W_n).$$

- (3) *Let \tilde{z} be the isomorphism between standard blocks given by the rotation (see [TA] for details):*

$$(5.1) \quad \tilde{z}: S_n(g_1, \dots, g_n; h_1, \dots, h_n) \rightarrow S_n(g_n, g_1, \dots, g_{n-1}; h_n, h_1, \dots, h_{n-1}).$$

Then \tilde{z}_ is given by:*

$$\begin{aligned} \text{Hom}_{\mathcal{C}}(\mathbf{1}, X_1 \otimes \dots \otimes X_n) &\rightarrow \text{Hom}_{\mathcal{C}}(\mathbf{1}, X_1 \otimes \dots \otimes X_n^{**}) \\ &\rightarrow \text{Hom}_{\mathcal{C}}(X_n^*, X_1 \otimes \dots \otimes X_{n-1}) \rightarrow \text{Hom}_{\mathcal{C}}(\mathbf{1}, X_n \otimes X_1 \otimes \dots \otimes X_{n-1}) \end{aligned}$$

where $X_i = {}^{h_i^{-1}}W_i$.

- (4) Let \tilde{b} be the braiding morphism between standard blocks (see [TA] for details):

$$\tilde{b}: S_3(g_1, g_2, g_3; h_1, h_2, h_3) \rightarrow S_3(g_1, g_2 g_3 g_2^{-1}, g_2; h_1, h_3 g_2^{-1}, h_2)$$

Then \tilde{b}_* is given by

$$\begin{aligned} \text{Hom}_{\mathcal{C}}(\mathbf{1}, X_1 \otimes X_2 \otimes X_3) &\xrightarrow{\text{id} \otimes R_{X_2, X_3}} \text{Hom}_{\mathcal{C}}(\mathbf{1}, X_1 \otimes {}^{g_2}X_3 \otimes X_2) \\ &= \text{Hom}_{\mathcal{C}}(\mathbf{1}, \otimes^{h_1^{-1}} W_1 \otimes \otimes^{g_2 h_3^{-1}} W_3 \otimes \otimes^{h_2^{-1}} W_2) \end{aligned}$$

where, as before, $X_i = \otimes^{h_i^{-1}} W_i$. Note that since $W_2 \in \mathcal{C}_{h_2 g_2 h_2^{-1}}$, $X_2 = \otimes^{h_2^{-1}} W_2 \in \mathcal{C}_{g_2}$. So by the definition of the braiding isomorphism of the fusion category, we need to twist X_3 by g_2 .

- (5) Let

$$\phi_x: S_n(g_1, \dots, g_n; h_1, \dots, h_n) \rightarrow S_n(xg_1x^{-1}, \dots, xg_nx^{-1}; h_1x^{-1}, \dots, h_nx^{-1})$$

be the isomorphism between standard blocks described in Lemma 3.6. Then $(\phi_x)_*$ is given by the following formula:

$$\begin{aligned} \text{Hom}_{\mathcal{C}}(\mathbf{1}, \otimes^{h_1^{-1}} V_1 \otimes \dots \otimes \otimes^{h_n^{-1}} V_n) &\rightarrow \text{Hom}_{\mathcal{C}}({}^x\mathbf{1}, \otimes^{xh_1^{-1}} V_1 \otimes \dots \otimes \otimes^{xh_n^{-1}} V_n) \\ &\rightarrow \text{Hom}_{\mathcal{C}}(\mathbf{1}, \otimes^{xh_1^{-1}} V_1 \otimes \dots \otimes \otimes^{xh_n^{-1}} V_n) \end{aligned}$$

We use the fact that ${}^x\mathbf{1} \cong \mathbf{1}$ and the action of the group G is a tensor functor.

This is not the full list of all the properties enjoyed by the correspondence between modular functors and equivariant categories; more properties will be seen from the construction.

Idea of the proof. Instead of proving a direct correspondence between G -equivariant modular functor and G -equivariant fusion category, we will introduce an intermediate object, which we will call G equivariant Moore-Seiberg data and then established the equivalence between these three notions:

$$(5.2) \quad \begin{array}{ccccc} G\text{-equivariant} & & G\text{-equivariant} & & G\text{-equivariant} \\ \text{modular functor} & \longleftrightarrow & \text{MS data} & \longleftrightarrow & \text{weakly fusion category} \end{array}$$

□

6. G -EQUIVARIANT MOORE-SEIBERG DATA

In this section we will introduce the intermediate object, the Moore-Seiberg data (MS for short); the goal of this is encoding the structure of a fusion category in terms of vector spaces $\langle W_1, \dots, W_n \rangle = \text{Hom}_{\mathcal{C}}(\mathbf{1}, W_1 \otimes \dots \otimes W_n)$ and suitable isomorphisms between such spaces.

For Moore-Seiberg data for $G = 1$ case, see [BK2, Section 5.3]. Here we need the extension of this concept to G -equivariant case.

Definition 6.1. Let \mathcal{C} be a G -equivariant abelian category (see Definition 2.1). Then G -equivariant Moore-Seiberg data (G -MS data for short) is the following collection of data:

Conformal blocks: For each $n \geq 0$ and $m_1, m_2, \dots, m_n \in G$ satisfying $m_1 m_2 \cdots m_n = 1$ we have a functor:

$$\langle \rangle: \mathcal{C}_{m_1} \boxtimes \cdots \boxtimes \mathcal{C}_{m_n} \rightarrow \mathcal{V}ec$$

where $\mathcal{V}ec$ denotes the category of finite-dimensional vector spaces.

Note: We can trivially extend this functor to a functor $\mathcal{C} \boxtimes \cdots \boxtimes \mathcal{C} \rightarrow \mathcal{V}ec$ by letting $\langle W_1, W_2, \dots, W_n \rangle = 0$ if $W_i \in m_i$, $m_1 m_2 \cdots m_n \neq 1$.

G -invariance: For each $g \in G$, we have functorial isomorphism

$$\phi_g: \langle V_1, V_2, \dots, V_n \rangle \longrightarrow \langle {}^g V_1, {}^g V_2, \dots, {}^g V_n \rangle.$$

satisfying $\phi_{gh} = \phi_h \phi_g$, $\phi_1 = \text{id}$.

Rotation isomorphism: Functorial isomorphism

$$Z: \langle V_1, V_2, \dots, V_n \rangle \rightarrow \langle V_n, V_1, \dots, V_{n-1} \rangle$$

Symmetric object: A symmetric G -invariant object $\mathcal{R} \in \mathcal{C} \boxtimes \mathcal{C}$ as in Definition 4.1.

Gluing isomorphism: For each $k, l \in \mathbb{Z}_+$, there exist functorial isomorphism

$$\mathcal{G}: \langle A_1, \dots, A_k, \mathcal{R}^1 \rangle \otimes \langle \mathcal{R}^2, B_1, \dots, B_l \rangle \rightarrow \langle A_1, \dots, A_k, B_1, \dots, B_l \rangle$$

(As before, $\mathcal{R}^1, \mathcal{R}^2$ should be understood as in (4.1)).

Commutativity isomorphism: For $A \in \mathcal{C}_g$ and $B \in \mathcal{C}_h$, we have functorial isomorphism

$$\sigma: \langle X, A, B \rangle \rightarrow \langle X, {}^g B, A \rangle.$$

Note: if $X \in \mathcal{C}_p$ then $\langle X, A, B \rangle$ is zero unless $pgh = 1$. But then ${}^g B \in \mathcal{C}_{ghg^{-1}}$ and the right hand side is zero unless we have $pghg^{-1}g = pgh = 1$; thus, if one side is zero for grading reasons, then so is the other.

These above data must satisfy the axioms formulated below: non-degeneracy, normalization, rotation axiom, associativity of G , symmetry of G , compatibility of ϕ , hexagon axiom, and Dehn twist axiom.

Before formulating the axioms, it would be convenient to define certain compositions of elementary gluing, rotation, and commutativity isomorphisms as follows.

Generalized gluing: For any $k, l, m \geq 0$, we define *generalized gluing* isomorphism

$$(6.1) \quad \begin{aligned} \mathcal{G}: \langle A_1, \dots, A_k, \mathcal{R}^1, C_1, \dots, C_m \rangle \otimes \langle \mathcal{R}^2, B_1, \dots, B_l \rangle \\ \rightarrow \langle A_1, \dots, A_k, B_1, \dots, B_l, C_1, \dots, C_m \rangle \end{aligned}$$

as the following composition:

$$\begin{aligned} & \langle A_1, \dots, A_k, \mathcal{R}^1, C_1, \dots, C_m \rangle \otimes \langle \mathcal{R}^2, B_1, \dots, B_l \rangle \\ & \xrightarrow{Z^m \otimes \text{id}} \langle C_1, \dots, C_m, A_1, \dots, A_k, \mathcal{R}^1 \rangle \otimes \langle \mathcal{R}^2, B_1, \dots, B_l \rangle \\ & \xrightarrow{\mathcal{G}} \langle C_1, \dots, C_m, A_1, \dots, A_k, B_1, \dots, B_l \rangle \\ & \xrightarrow{Z^{-m}} \langle A_1, \dots, A_k, B_1, \dots, B_l, C_1, \dots, C_m \rangle \end{aligned}$$

Generalized commutativity: For any $k, l \geq 0$, we define the generalized commutativity isomorphisms

$$(6.2) \quad \sigma: \langle A_1, \dots, A_k, X, Y, B_1, \dots, B_l \rangle \rightarrow \langle A_1, \dots, A_k, {}^p Y, X, B_1, \dots, B_l \rangle$$

where $X \in \mathcal{C}_p$ and $Y \in \mathcal{C}_q$, as the following composition

$$\begin{aligned} \langle A_1, \dots, A_k, X, Y, B_1, \dots, B_l \rangle &\xrightarrow{\mathcal{G}^{-1}} \langle A_1, \dots, A_k, \mathcal{R}^1, B_1, \dots, B_l \rangle \otimes \langle \mathcal{R}^2, X, Y \rangle \\ &\xrightarrow{\text{id} \otimes \sigma} \langle A_1, \dots, A_k, \mathcal{R}^1, B_1, \dots, B_l \rangle \otimes \langle \mathcal{R}^2, {}^pY, X \rangle \\ &\xrightarrow{\mathcal{G}} \langle A_1, \dots, A_k, {}^pY, X, B_1, \dots, B_l \rangle \end{aligned}$$

(here \mathcal{G} is the generalized gluing (6.1)).

From now on we will denote by σ both the generalized commutativity isomorphisms and the usual commutativity isomorphisms.

One can define even more general isomorphisms; however, it won't be necessary for our purposes.

Now we are ready to formulate the axioms of the MS data.

Non-degeneracy: For each object $X \in \mathcal{C}$, there exists an object $X \in \mathcal{C}$ so that $\langle X, V \rangle \neq 0$.

Normalization: For $n = 0$

$$\langle \rangle: \mathcal{V}ec \rightarrow \mathcal{V}ec$$

is the identity functor (as before, $\mathcal{V}ec$ denotes the category of finite-dimensional vector spaces).

Associativity of \mathcal{G} : Let $\mathcal{R}, \tilde{\mathcal{R}}$ be two copies of \mathcal{R} . Then the diagram in Figure 4 is commutative.

$$\begin{array}{ccc} \langle A_1, \dots, A_k, \mathcal{R}^1 \rangle \otimes \langle \mathcal{R}^2, B_1, \dots, B_l, \tilde{\mathcal{R}}^2 \rangle \otimes \langle \tilde{\mathcal{R}}^2, C_1, \dots, C_p \rangle & & \\ \mathcal{G} \otimes \text{id} \swarrow & & \searrow \text{id} \otimes \mathcal{G} \\ \langle A_1, \dots, A_k, B_1, \dots, B_l, \tilde{\mathcal{R}}^2 \rangle \otimes \langle \tilde{\mathcal{R}}^2, C_1, \dots, C_p \rangle & & \langle A_1, \dots, A_k, \mathcal{R}^1 \rangle \otimes \langle \mathcal{R}^2, B_1, \dots, B_l, C_1, \dots, C_p \rangle \\ \mathcal{G} \swarrow & & \searrow \mathcal{G} \\ \langle A_1, \dots, A_k, B_1, \dots, B_l, C_1, \dots, C_p \rangle & & \end{array}$$

FIGURE 4. Associativity axiom for \mathcal{G}

Rotation axiom: The isomorphism of the vector spaces

$$Z^n: \langle V_1, \dots, V_n \rangle \rightarrow \langle V_1, \dots, V_n \rangle$$

is equal to identity.

Symmetry of \mathcal{G} : Again let $\mathcal{R} = \mathcal{R}^1 \boxtimes \mathcal{R}^2$ and P be the usual isomorphism between the vector spaces $A \otimes B \rightarrow B \otimes A$ given by $P(a \otimes b) = b \otimes a$. Then the diagram in Figure 5 is commutative.

Compatibility of ϕ : The functorial isomorphisms ϕ_g must be compatible with all the other isomorphisms. More precisely we have the following:

$$\begin{array}{ccc}
\langle A_1, \dots, A_k, \mathcal{R}^1 \rangle \otimes \langle \mathcal{R}^2, B_1, \dots, B_l \rangle & \xrightarrow{\mathcal{G}} & \langle A_1, \dots, A_k, B_1, \dots, B_l \rangle \\
\downarrow Z \otimes Z^{-1} & & \downarrow Z^t \\
\langle \mathcal{R}^1, A_1, \dots, A_k \rangle \otimes \langle B_1, \dots, B_l, \mathcal{R}^2 \rangle & & \\
\downarrow & & \\
\langle B_1, \dots, B_l, \mathcal{R}^2 \rangle \otimes \langle \mathcal{R}^1, A_1, \dots, A_k \rangle & \xrightarrow{\mathcal{G}} & \langle B_1, \dots, B_l, A_1, \dots, A_k \rangle
\end{array}$$

FIGURE 5. Symmetry of \mathcal{G} axiom

- (1) ϕ must be compatible with rotation isomorphism, i.e. the following diagram is commutative:

$$\begin{array}{ccc}
\langle W_1, \dots, W_n \rangle & \xrightarrow{Z} & \langle W_n, W_1, \dots, W_{n-1} \rangle \\
\downarrow \phi_g & & \downarrow \phi_g \\
\langle {}^g W_1, \dots, {}^g W_n \rangle & \xrightarrow{Z} & \langle {}^g W_n, {}^g W_1, \dots, {}^g W_{n-1} \rangle
\end{array}$$

- (2) ϕ must be compatible with the gluing isomorphism, i.e. for each $k, l \in \mathbb{Z}_+$ the following diagram must be commutative

$$\begin{array}{ccc}
\langle A_1, \dots, A_k, \mathcal{R}^1 \rangle \otimes \langle \mathcal{R}^2, B_1, \dots, B_l \rangle & \xrightarrow{\mathcal{G}} & \langle A_1, \dots, A_k, B_1, \dots, B_l \rangle \\
\downarrow \phi_g \otimes \phi_g & & \downarrow \phi_g \\
\langle {}^g A_1, \dots, {}^g A_k, {}^g \mathcal{R}^1 \rangle \otimes \langle {}^g \mathcal{R}^2, {}^g B_1, \dots, {}^g B_l \rangle & & \\
\downarrow & & \\
\langle {}^g A_1, \dots, {}^g A_k, \mathcal{R}^1 \rangle \otimes \langle \mathcal{R}^2, {}^g B_1, \dots, {}^g B_l \rangle & \xrightarrow{\mathcal{G}} & \langle {}^g A_1, \dots, {}^g A_k, {}^g B_1, \dots, {}^g B_l \rangle
\end{array}$$

The diagram above uses the isomorphism ${}^g \mathcal{R}^1 \boxtimes {}^g \mathcal{R}^2 = \mathcal{R}^1 \boxtimes \mathcal{R}^2$.

- (3) ϕ must be compatible with commutativity isomorphism, i.e. for any $A \in \mathcal{C}_p$ and $B \in \mathcal{C}_q$ the following diagram must be commutative.

$$\begin{array}{ccc}
\langle X, A, B \rangle & \xrightarrow{\sigma} & \langle X, {}^p B, A \rangle \\
\downarrow \phi_g & & \downarrow \phi_g \\
\langle {}^g X, {}^g A, {}^g B \rangle & \xrightarrow{\sigma} & \langle {}^g X, {}^{gp} B, {}^g A \rangle
\end{array}$$

Hexagon axiom: For any $A \in \mathcal{C}_p, B \in \mathcal{C}_q$ and $C \in \mathcal{C}_r$, the diagram below and a similar diagram with σ replaced by σ^{-1} must be commutative

$$\begin{array}{ccc}
\langle X, A, B, C \rangle & \xrightarrow{\sigma_{A,BC}} & \langle X, {}^p B, {}^p C, A \rangle \\
\searrow \sigma_{A,B} & & \nearrow \sigma_{A,C} \\
& \langle X, {}^p B, A, C \rangle &
\end{array}$$

where $\sigma_{A,B}$, $\sigma_{A,C}$ are generalized braidings defined by (6.2) and $\sigma_{A,BC}$ is defined as the following composition

$$\begin{aligned} \langle X, A, B, C \rangle &\rightarrow \langle X, A, \mathcal{R}^1 \rangle \otimes \langle \mathcal{R}^2, B, C \rangle \\ &\rightarrow \langle X, {}^p\mathcal{R}^1, A \rangle \otimes \langle \mathcal{R}^2, B, C \rangle \rightarrow \langle X, \mathcal{R}^1, A \rangle \otimes \langle {}^{p^{-1}}\mathcal{R}^2, B, C \rangle \\ &\rightarrow \langle X, \mathcal{R}^1, A \rangle \otimes \langle \mathcal{R}^2, {}^pB, {}^pC \rangle \rightarrow \langle X, {}^pB, {}^pC, A \rangle \end{aligned}$$

Dehn twist axiom: Let $A \in \mathcal{C}_p$ and $B \in \mathcal{C}_q$ with $pq = 1$. Then the Dehn twist axiom is the commutativity of the diagram in Figure 6. Note that if $pq \neq 1$ then all the vector spaces in Figure 6 will be zero and the diagram trivially commutes.

$$\begin{array}{ccccc} \langle A, B \rangle & \xrightarrow{\sigma} & \langle {}^pB, A \rangle & & \\ & \downarrow z & & \searrow z & \\ & \langle B, A \rangle & \xrightarrow{\sigma} & \langle {}^qA, B \rangle & \nearrow \varphi \\ & & & & \langle A, {}^pB \rangle \end{array}$$

FIGURE 6. Dehn twist axiom for G -MS data.

7. G -MS DATA AND G -FUSION CATEGORIES

We will divide the proof of the main theorem into two steps, first relating the notion of G -MS data with G -equivariant fusion categories, and then relating G -MS data with the G -modular functor. In this section we do the first step, showing the equivalence of G -equivariant (weakly) fusion category and G -equivariant MS data.

Theorem 7.1. (1) *If we have a structure of G equivariant (weakly) fusion category on \mathcal{C} then from this we can create a G equivariant MS data on \mathcal{C} .*
(2) *Conversely, given a G equivariant MS data on \mathcal{C} , we can define a structure of G equivariant weakly fusion category on \mathcal{C} .*
(3) *The above two constructions are inverse of each other.*

The proof of this theorem is given below. For the most part, it is parallel to the proof in $G = \{1\}$ case, given in [BK2, Section 5.3]; thus we will only provide detailed explanations of the steps which are new to G -equivariant case.

7.1. From fusion categories to G -MS data. Assume that \mathcal{C} has a structure of G equivariant (weakly) fusion category. We define the G equivariant MS data as follows

Conformal blocks: Let $W_1 \in \mathcal{C}_{m_1}, \dots, W_n \in \mathcal{C}_{m_n}$, where $m_1 m_2 \dots m_n = 1$. Then define

$$(7.1) \quad \langle W_1, W_2, \dots, W_n \rangle = \text{Hom}(\mathbf{1}, W_1 \otimes \dots \otimes W_n)$$

ϕ -axiom: For $g \in G$, define

$$\phi_g: \langle W_1, W_2, \dots, W_n \rangle \rightarrow \langle {}^gW_1, {}^gW_2, \dots, {}^gW_n \rangle$$

by

$$\text{Hom}(\mathbf{1}, W_1 \otimes \dots \otimes W_n) \xrightarrow{R_g} \text{Hom}({}^g\mathbf{1}, {}^g(W_1 \otimes \dots \otimes W_n)) \rightarrow \text{Hom}(\mathbf{1}, {}^gW_1 \otimes \dots \otimes {}^gW_n)$$

Here R_g is the action of the element $g \in G$. Since R_g is by definition a tensor functor, we have canonical isomorphisms ${}^g\mathbf{1} = \mathbf{1}$ and ${}^g(W_1 \otimes \cdots \otimes W_n) = {}^gW_1 \otimes \cdots \otimes {}^gW_n$.

Rotation isomorphism: Define the rotation isomorphism

$$Z: \langle W_1, W_2, \dots, W_n \rangle \rightarrow \langle W_n, W_1, \dots, W_{n-1} \rangle$$

by

$$(7.2) \quad \begin{aligned} \text{Hom}(\mathbf{1}, W_1 \otimes W_2 \otimes \cdots \otimes W_n) &\rightarrow \text{Hom}(\mathbf{1}, W_1 \otimes W_2 \otimes \cdots \otimes W_{n-1} \otimes W_n^{**}) \\ &\rightarrow \text{Hom}(W_n^*, W_1 \otimes \cdots \otimes W_{n-1}) \rightarrow \text{Hom}(\mathbf{1}, W_n \otimes W_1 \otimes \cdots \otimes W_{n-1}) \end{aligned}$$

Here we use the rigidity isomorphisms, balancing isomorphism $\delta: V \rightarrow V^{**}$, and the fact that in any weakly rigid fusion category we have an isomorphism $\text{Hom}(\mathbf{1}, X \otimes W^{**}) \simeq \text{Hom}(W^*, X)$ (see [BK2]).

Symmetric object: The symmetric object \mathcal{R} is defined by

$$\mathcal{R} = V_i \boxtimes V_i^*$$

where $\{V_i\}_{i \in I}$ are representatives of the isomorphism classes of simple objects. The fact that it is independent of the choice of representatives and symmetry of \mathcal{R} are shown in the same way as in [BK2].

To show G -invariance, note that since R_g is a tensor functor, it must take a simple object to a simple object. Thus, for every i we can choose an isomorphism $\psi_g: R_g(V_i) \rightarrow V_j$ for some $j \in I$. Now define isomorphism

$$(R_g \boxtimes R_g)\mathcal{R} = \bigoplus R_g(V_i) \boxtimes R_g(V_i^*) \simeq \bigoplus R_g(V_i) \boxtimes (R_g(V_i))^* \xrightarrow{\psi_g \boxtimes (\psi_g^*)^{-1}} \bigoplus V_j \boxtimes V_j^*$$

It is easy to see that this isomorphism does not depend on the choice of ψ and is compatible with the symmetry $\sigma: \mathcal{R} \rightarrow \mathcal{R}^p$.

Gluing isomorphism: The gluing isomorphism

$$\langle A_1, \dots, A_k, \mathcal{R}^1 \rangle \otimes \langle \mathcal{R}^2, B_1, \dots, B_l \rangle \rightarrow \langle A_1, \dots, A_k, B_1, \dots, B_l \rangle$$

is defined by

$$(7.3) \quad \begin{aligned} &\bigoplus \text{Hom}(\mathbf{1}, A_1 \otimes \cdots \otimes A_k \otimes V_i^*) \otimes \text{Hom}(\mathbf{1}, V_i \otimes B_1 \otimes \cdots \otimes B_l) \\ &\cong \bigoplus \text{Hom}(\mathbf{1}, A_1 \otimes \cdots \otimes A_k \otimes V_i^*) \otimes \text{Hom}(V_i^*, B_1 \otimes \cdots \otimes B_l) \\ &\cong \bigoplus \text{Hom}(\mathbf{1}, A_1 \otimes \cdots \otimes A_k \otimes B_1 \otimes \cdots \otimes B_l) \end{aligned}$$

Commutativity isomorphism: Let $A \in \mathcal{C}_p$ and $B \in \mathcal{C}_q$. Define the commutativity isomorphism

$$\sigma: \langle X, A, B \rangle \rightarrow \langle X, {}^pB, A \rangle$$

by

$$(7.4) \quad \text{Hom}(\mathbf{1}, X \otimes A \otimes B) \xrightarrow{\text{id} \otimes R_{A,B}} \text{Hom}(\mathbf{1}, X \otimes {}^pB \otimes A).$$

Then the above data satisfies the definition of G -MS data. Indeed, it can easily be shown that the isomorphisms defined above satisfy all the axioms of MS-data; the easiest way to do this is to use the technique of using appropriately marked ribbon graphs to represent morphisms in a G -equivariant fusion category (see [T2], [K1]).

7.2. From G -MS data to G -fusion categories. Now we are assuming that we have a G -equivariant Moore-Seiberg data and from this we want to construct a structure of G equivariant weakly fusion category on \mathcal{C} . This construction is parallel to the construction in the case of $G = \{1\}$ given in [BK2, Section 5.3].

We will construct the structure of weakly fusion category step by step as follows:

Duality: Define the functor $*$ by

$$\langle V, X \rangle = \text{Hom}(V^*, X)$$

Remark 7.2. Every object V of \mathcal{C} is completely determined by the functor $\langle V, \cdot \rangle$.

The arguments in [BK2, Section 5.3], without any changes, show that $*$ is an antiequivalence of categories, and that one has a canonical isomorphism $\mathcal{R} \cong \bigoplus V_i \boxtimes V_i^*$.

Construction of tensor product: As in [BK2], define tensor product functor by

$$\langle X, A \otimes B \rangle = \langle X, A, B \rangle$$

Note that for $X \in \mathcal{C}_r, A \in \mathcal{C}_p, B \in \mathcal{C}_q$, one has $\langle X, A, B \rangle = 0$ unless $rpq = 1$; this implies that $A \otimes B \in \mathcal{C}_{pq}$.

The same arguments as in [BK2, Section 5.3] show that this tensor product has a canonical associativity isomorphism; moreover, we have canonical isomorphisms

$$\langle X, A_1 \otimes \cdots \otimes A_n \rangle \cong \langle X, A_1, \dots, A_n \rangle.$$

Construction of unit object: We define the unit object by

$$\langle \mathbf{1}, X \rangle = \langle X \rangle$$

Again, the same argument as in [BK2, Section 5.3], with no changes at all, shows that so defined $\mathbf{1}$ is a unit object with respect to previously defined tensor product, and that we have a canonical isomorphism $\mathbf{1}^* \cong \mathbf{1}$.

Construction of commutativity isomorphism: Let $A \in \mathcal{C}_p$ and $B \in \mathcal{C}_q$. We define

$$R_{A,B}: A \otimes B \rightarrow {}^p B \otimes A$$

as the following composition:

$$\langle X, A \otimes B \rangle = \langle X, A, B \rangle \xrightarrow{\sigma} \langle X, {}^p B, A \rangle = \langle X, {}^p B \otimes A \rangle$$

Here σ is the commutativity isomorphism of G equivariant MS data.

Easy explicit computation shows that the hexagon axiom of fusion category is exactly equivalent to the hexagon axiom of G equivariant MS data.

Construction of balancing isomorphisms: We know that having functorial isomorphism $\delta: V \rightarrow V^{**}$ is equivalent to having functorial isomorphisms (or twist) $\theta_V: V \rightarrow {}^g V$ for $V \in \mathcal{C}_g$ satisfying certain conditions (see Lemma 2.4).

To define θ , recall the generalized commutativity isomorphisms defined by (6.2). In particular, letting $k, l = 0$, we get a commutativity isomorphism

$$\sigma: \langle A, B \rangle \rightarrow \langle {}^p B, A \rangle, \quad A \in \mathcal{C}_p, B \in \mathcal{C}_q.$$

Now define the twist functor $\theta_V: V \rightarrow {}^g V, V \in \mathcal{C}_g$ as follows. For any $X \in \mathcal{C}$, consider the composition

$$\langle V, X \rangle \xrightarrow{\sigma^{-1}} \langle X, {}^g V \rangle \xrightarrow{Z} \langle {}^g V, X \rangle.$$

This gives the functorial isomorphism between the functor $\langle V, . \rangle$ and $\langle {}^g V, . \rangle$ which in turn gives the twist θ_V .

All the required properties of θ , listed in Lemma 2.4, now easily follow from the properties of commutativity and associativity morphisms in the definition of MS data.

8. MS DATA AND MODULAR FUNCTOR

In this section we do the second step of the proof, showing the equivalence of G equivariant genus zero modular functor and G equivariant MS data.

Theorem 8.1. *Let \mathcal{C} be a semisimple G -equivariant abelian category with a finite number of equivalence classes of simple objects.*

- (1) *If we have a non-degenerate G equivariant \mathcal{C} extended genus zero modular functor then we can define a G equivariant MS data on \mathcal{C}*
- (2) *Conversely, given a G equivariant MS data on \mathcal{C} , we can define a non-degenerate G equivariant \mathcal{C} extended genus zero modular functor.*
- (3) *The above two constructions are inverse to each other.*

The proof of this theorem is given in two subsections below.

8.1. From G -MS data to G -MF. Now we are assuming that we have a G equivariant MS data on \mathcal{C} and from this we want to construct a G equivariant \mathcal{C} extended genus zero modular functor. The construction is similar to the one given in [BK2] in $G = \{1\}$ case: we first define the modular functor on standard blocks. Since any G cover is isomorphic to gluing of several standard blocks (this identification is not unique; in fact there are infinitely many parameterization of a given G cover), this will give us the modular functor on parametrized G -covers. After this, we show that the modular functor spaces obtained from any two parameterizations of the same G cover are canonically isomorphic.

Definition 8.2. Given $W_i \in \mathcal{C}_{h_i g_i h_i^{-1}}$ for $i = 1 \dots n$, we define

$$\tau[S_n(g_1, \dots, g_n; h_1, \dots, h_n; W_1, \dots, W_n)] = \langle X_1, \dots, X_n \rangle$$

where for brevity we denoted

$$X_i = h_i^{-1} W_i.$$

Remark 8.3. Note that monodromy around i th boundary component of $S_n(g_1, \dots, g_n; h_1, \dots, h_n)$ is $m_i = h_i g_i^{-1} h_i^{-1}$, so we have $W_i \in \mathcal{C}_{m_i^{-1}}$, as required in the definition of the modular functor. Note also that $X_i \in \mathcal{C}_{g_i}$, so condition $g_1 \dots g_n = 1$ given by definition of standard block matches the condition required for $\langle X_1, \dots, X_n \rangle$ to be non-zero.

Next, let $\tilde{\Sigma}$ be a G -cover and $W_a \in \mathcal{C}_{m_a^{-1}}$, $a \in A(\Sigma)$ — a collection of objects assigned to boundary components of Σ . Let

$$f: \tilde{\Sigma} \rightarrow S_{n_1}(\mathbf{g}^1, \mathbf{h}^1) \bigsqcup_{\text{glued}} \dots \bigsqcup_{\text{glued}} S_{n_k}(\mathbf{g}^k, \mathbf{h}^k)$$

be a parameterization of $\tilde{\Sigma}$ (see Definition 3.9). For each connected component Σ_i of $\Sigma \setminus \text{cuts}$, let us assign an object of \mathcal{C} to each boundary component of Σ_i ,

by putting W_a on the boundary of Σ and a copy of \mathcal{R} on each cut (i.e., assigning \mathcal{R}^1 to the component on one side of the cut and \mathcal{R}^2 on the other side; since \mathcal{R} is symmetric, choice of the side is not important).

Then we define the modular functor for parameterized G -covers by

$$\tau[\tilde{\Sigma}, \{W_a\}, f] = \bigotimes_i \tau[S_{n_i}(\mathbf{g}^i, \mathbf{h}^i)]$$

with the choice of objects as explained above. This defines a modular functor for parameterized surfaces; by definition, it satisfies the gluing axiom. Now we need to identify the spaces $\tau[\tilde{\Sigma}, \{W_a\}, f]$ for different parameterizations.

To do so, we use the same strategy used in $G = \{1\}$ case in [BK2]. Namely, recall the complex $M(\tilde{\Sigma}, \Sigma)$ from [TA]; vertices of this complex are exactly different parameterizations of $\tilde{\Sigma}$, edges are certain “simple moves” between parameterizations, and 2-cells describe relations. The main result of the paper [TA] is that under these basic moves and relations the complex $M(\tilde{\Sigma}, \Sigma)$ is connected and simply connected.

Now, for any simple move **E** connecting parameterizations f, g , we will define a functorial isomorphism $E: \tau[\tilde{\Sigma}, \{W_a\}, f] \rightarrow \tau[\tilde{\Sigma}, \{W_a\}, g]$ as follows. Recall from [TA] that simple moves are **Z** (rotation), **B** (braiding), **F** (fusion, or gluing), **P** (which is related to isomorphism ϕ_x between standard blocks), **T** (change of marked points on a cut).

Z move: If f is the parameterization

$$f: \tilde{\Sigma} \rightarrow S_n(g_1, g_2, \dots, g_n; h_1, h_2, \dots, h_n)$$

then $\mathbf{Z}(f)$ is the parameterization

$$\mathbf{Z}(f): \tilde{\Sigma} \rightarrow S_n(g_n, g_1, \dots, g_{n-1}; h_n, h_1, \dots, h_{n-1})$$

Now the corresponding map between $\tau[\tilde{\Sigma}, \{W_a\}, f]$ and $\tau[\tilde{\Sigma}, \{W_a\}, \mathbf{Z}(f)]$ is given by the rotation isomorphism of G -equivariant MS data which we also denoted by Z :

$$Z: \langle X_1, \dots, X_n \rangle \rightarrow \langle X_n, X_1, \dots, X_{n-1} \rangle$$

B move: Now let f be the parameterization

$$f: \tilde{\Sigma} \rightarrow S_3(g_1, g_2, g_3; h_1, h_2, h_3)$$

then $\mathbf{B}(f)$ is the parameterization

$$\mathbf{B}(f): \tilde{\Sigma} \rightarrow S_3(g_1, g_2 g_3 g_2^{-1}, g_2; h_1, h_3 g_2^{-1}, h_2)$$

Then the corresponding map between $\tau[\tilde{\Sigma}, \{W_a\}, f]$ and $\tau[\tilde{\Sigma}, \{W_a\}, \mathbf{B}(f)]$ is given by the commutativity isomorphism of G equivariant MS data which we denoted by σ :

$$\sigma: \langle {}^{h_1^{-1}}W_1, {}^{h_2^{-1}}W_2, {}^{h_3^{-1}}W_3 \rangle \rightarrow \langle {}^{h_1^{-1}}W_1, {}^{g_2 h_3^{-1}}W_3, {}^{h_2^{-1}}W_2 \rangle$$

Remark 8.4. Since $W_2 \in \mathcal{C}_{h_2 g_2 h_2^{-1}}$, we must have ${}^{h_2^{-1}}W_2 \in \mathcal{C}_{g_2}$. Thus from the definition of the commutativity isomorphism, we need to twist ${}^{h_3^{-1}}W_3$ by g_2 .

F move: Let f be the parameterization

$$f: \tilde{\Sigma} \rightarrow S_{k+1}(\mathbf{g}, \mathbf{h}) \bigsqcup_{k+1,1} S_{l+1}(\mathbf{g}', \mathbf{h}')$$

Here we write $\mathbf{g} = g_1, \dots, g_{k+1}$, etc. for simplicity. Assume additionally that $h_{k+1} = h'_1$. Then $\mathbf{F}(f)$ is the parameterization given by

$$\mathbf{F}(f): \tilde{\Sigma} \rightarrow S_{k+l}(\mathbf{g}'', \mathbf{h}'')$$

where

$$\begin{aligned} \mathbf{g}'' &= (g_1, \dots, g_k, g'_2, \dots, g'_{l+1}) \\ \mathbf{h}'' &= (h_1, \dots, h_k, h'_2, \dots, h'_{l+1}) \end{aligned}$$

Then the corresponding map between $\tau[\tilde{\Sigma}, \{W_a\}, f]$ and $\tau[\tilde{\Sigma}, \{W_a\}, \mathbf{F}(f)]$ is given by the gluing isomorphism of G equivariant MS data which we denote by \mathcal{G}

$$\mathcal{G}: \langle X_1, \dots, X_k, \mathcal{R}^1 \rangle \otimes \langle \mathcal{R}^2, Y_1, \dots, Y_l \rangle \rightarrow \langle X_1, \dots, X_k, Y_1, \dots, Y_l \rangle$$

P_x move: Let $x \in \mathcal{G}$, and let f be the following parameterization

$$f: \tilde{\Sigma} \rightarrow S_n(g_1, \dots, g_n; h_1, \dots, h_n)$$

then $\mathbf{P}_x(f)$ is the parameterization

$$\mathbf{P}_x(f): \tilde{\Sigma} \rightarrow S_n(xg_1x^{-1}, xg_2x^{-1}, \dots, xg_nx^{-1}; h_1x^{-1}, h_2x^{-1}, \dots, h_nx^{-1})$$

Then the corresponding map between $\tau[\tilde{\Sigma}, \{W_a\}, f]$ and $\tau[\tilde{\Sigma}, \{W_a\}, \mathbf{P}_x(f)]$ is given by the ϕ_x isomorphism as defined in ϕ axiom of G equivariant MS data:

$$\phi_x: \langle {}^{h_1^{-1}}W_1, \dots, {}^{h_n^{-1}}W_n \rangle \rightarrow \langle {}^{xh_1^{-1}}W_1, \dots, {}^{xh_n^{-1}}W_n \rangle$$

T move: Let f be the parameterization

$$f: \tilde{\Sigma} \rightarrow S_n(\dots, x; \dots, y) \bigsqcup_{n,1} S_m(x^{-1}, \dots; y, \dots)$$

where we avoid writing all g_i and h_i for simplicity and only write the labels for the boundary we want to glue. Then **T** move replaces the label $y \in G$ by another label $z \in G$:

$$\mathbf{T}(f): \tilde{\Sigma} \rightarrow S_n(\dots, x; \dots, z) \bigsqcup_{n,1} S_m(x^{-1}, \dots; z, \dots)$$

(geometrically, it means that we are changing the choice of marked point on the corresponding cut). Then the corresponding map between $\tau[\tilde{\Sigma}, \{W_a\}, f]$ and $\tau[\tilde{\Sigma}, \{W_a\}, \mathbf{T}(f)]$ is given by the symmetry of \mathcal{R} under the action of G , which is part of the definition of G -equivariant modular functor:

$$\langle \dots, {}^{y^{-1}}\mathcal{R}^1 \rangle \otimes \langle {}^{y^{-1}}\mathcal{R}^2, \dots \rangle \rightarrow \langle \dots, {}^{z^{-1}}\mathcal{R}^1 \rangle \otimes \langle {}^{z^{-1}}\mathcal{R}^2, \dots \rangle$$

is given by identifying both ${}^{y^{-1}}\mathcal{R}^1 \boxtimes {}^{y^{-1}}\mathcal{R}^2$ and ${}^{z^{-1}}\mathcal{R}^1 \boxtimes {}^{z^{-1}}\mathcal{R}^2$ with $\mathcal{R}^1 \boxtimes \mathcal{R}^2$

So far we have translated all our simple moves from the language of parameterizations of G -covers to the language of G -equivariant MS data. Since any two parametrizations f and g can be connected by a sequence of simple moves (the complex $M(\tilde{\Sigma}, \Sigma)$ is connected), we can construct an isomorphism of the corresponding vector spaces, $\tau[\tilde{\Sigma}, \{W_a\}, f]$ and $\tau[\tilde{\Sigma}, \{W_a\}, g]$. Now we need to show that we get the same isomorphism of the vector spaces independent of the choice

of path (composed of simple moves). Since the complex $M(\tilde{\Sigma}, \Sigma)$ is simply connected, it is enough to show that all the basic relations or 2-cells of $M(\tilde{\Sigma}, \Sigma)$ can be translated to the corresponding axioms of G -equivariant MS data.

The following is the full list of all relations (2-cells) in the complex $M(\tilde{\Sigma}, \Sigma)$; precise statements of the relations can be found in [TA].

- \mathbf{P}_x relation
- $\mathbf{P} - \mathbf{F}$ relation
- \mathbf{Z} relation
- \mathbf{B} relation
- \mathbf{T} relation
- Rotation axiom
- Commutativity of disjoint union
- Symmetry of \mathbf{F} move
- Associativity of cuts
- Cylinder axiom
- Braiding axiom
- Dehn twist axiom

We now show how each of these relations follows from the axioms of MS-data.

\mathbf{P}_x relation: The relation $\mathbf{P}_x \mathbf{Z} = \mathbf{Z} \mathbf{P}_x$ corresponds to the compatibility of ϕ with rotation axiom.

The relation $\mathbf{P}_x \mathbf{B} = \mathbf{B} \mathbf{P}_x$ corresponds to the compatibility of ϕ with commutativity isomorphism.

The relation $\mathbf{P}_x \mathbf{F}_{c,y} = \mathbf{F}_{c,yx^{-1}}(\mathbf{P}_x \sqcup \mathbf{P}_x)$ corresponds to the compatibility of ϕ with gluing isomorphism.

The relation $\mathbf{P}_x \mathbf{P}_y = \mathbf{P}_{xy}$ corresponds to the ϕ relation of the definition of MS data.

Rotation axiom: The rotation axiom, $\mathbf{Z}^n = \text{id}$, of the complex $M(\tilde{\Sigma}, \Sigma)$ corresponds to the rotations axiom $Z^n = \text{id}$ of the MS data

Symmetry of \mathbf{F} move: Symmetry of the \mathbf{F} move in $M(\tilde{\Sigma}, \Sigma)$ corresponds to the symmetry of gluing isomorphism, \mathcal{G} , in MS data

Associativity of cuts: Associativity of cuts of $M(\tilde{\Sigma}, \Sigma)$ corresponds to the associativity of gluing isomorphism, \mathcal{G} , in MS data

Braiding axiom: Braiding axioms of $M(\tilde{\Sigma}, \Sigma)$ corresponds to the hexagon axiom in MS data

Dehn twist axiom: Dehn twist axioms of $M(\tilde{\Sigma}, \Sigma)$ corresponds to the Dehn twist axiom in MS data.

Cylinder axiom: Cylinder axiom of $M(\tilde{\Sigma}, \Sigma)$ follows from functoriality of all the isomorphisms in the definition of MS data.

We leave it to the reader to supply the details of the above construction.

Thus, we have defined, for any G -cover $\tilde{\Sigma}$ and a parametrization f , a vector space $\tau[\tilde{\Sigma}, \{W_a\}, f]$ and have shown that for any two parametrizations, there is a canonical isomorphism between the vector spaces $\tau[\tilde{\Sigma}, \{W_a\}, f]$ and $\tau[\tilde{\Sigma}, \{W_a\}, g]$. Now the same arguments as in $G = \{1\}$ case (see [BK2]) show that this allows us to define a vector space $\tau[\tilde{\Sigma}, \{W_a\}]$, independent of parametrization, thus giving us the G -equivariant genus zero modular functor. Readers can easily show that the G -equivariant genus zero modular functor defined this way satisfies all the axioms of modular functor.

8.2. From G -MF to G -MS data. We assume that we are given a \mathcal{C} extended genus zero modular functor.

We want to create a G equivariant MS data on \mathcal{C} . To do this we define the following

Conformal blocks: Given $n \geq 0$ and $W_1 \in \mathcal{C}_{m_1} \dots, W_n \in \mathcal{C}_{m_n}$ satisfying $m_1 \dots m_n = 1$, we define

$$(8.1) \quad \langle W_1, \dots, W_n \rangle = \tau [S_n(m_1, \dots, m_n; 1, \dots, 1; [W_1, \dots, W_n])]$$

where $S_n(m_1, \dots, m_n; 1, \dots, 1)$ is the standard block defined in Definition 3.5.

ϕ **axiom:** For each $g \in G$ we define a functorial isomorphism $\phi_g: \langle W_1, \dots, W_n \rangle \rightarrow \langle {}^g W_1, \dots, {}^g W_n \rangle$ as the following composition:

$$\begin{aligned} & \tau [S_n(m_1, \dots, m_n; 1, \dots, 1; [W_1, \dots, W_n])] \\ & \xrightarrow{(\phi_g)_*} \tau [S_n(gm_1g^{-1}, \dots, gm_ng^{-1}; g^{-1}, \dots, g^{-1}; [W_1, \dots, W_n])] \\ & \xrightarrow{T_{g, \dots, g}} \tau [S_n(gm_1g^{-1}, \dots, gm_ng^{-1}; 1, \dots, 1; [{}^g W_1, \dots, {}^g W_n])] \end{aligned}$$

Here ϕ_g is the morphism between standard blocks described in Lemma 3.6, and T_g is as in the definition of the modular functor (see eq. (4.3)).

Rotation axiom: The rotation isomorphism

$$Z: \langle W_1, \dots, W_n \rangle \rightarrow \langle W_n, W_1, \dots, W_{n-1} \rangle$$

is given by $Z = (\tilde{z})_*$, where \tilde{z} is the rotation homeomorphism of the standard block (see eq. (5.1)).

Symmetric object: The symmetric object \mathcal{R} directly comes from the definition of G equivariant modular functor.

Gluing isomorphism: Let $A_i \in \mathcal{C}_{p_i}$ and $B_j \in \mathcal{C}_{q_j}$, where $i = 1 \dots k$ and $j = 1 \dots l$ and they satisfy $p_1 \dots p_k q_1 \dots q_l = 1$. Then the gluing isomorphism

$$\mathcal{G}: \langle A_1, \dots, A_k, \mathcal{R}^1 \rangle \otimes \langle \mathcal{R}^2, B_1, \dots, B_l \rangle \rightarrow \langle A_1, \dots, A_k, B_1, \dots, B_l \rangle$$

is given by the $(\tilde{\alpha}_{k,l})_*$, where

$$\begin{aligned} & \tilde{\alpha}_{k,l}: S_{k+1}(p_1, \dots, p_k, x; 1, \dots, 1) \bigsqcup_{k,1} S_{l+1}(x^{-1}, q_1, \dots, q_l; 1, \dots, 1) \\ & \rightarrow S_{k+l}(p_1, \dots, p_k, q_1, \dots, q_l; 1, \dots, 1) \end{aligned}$$

is the homeomorphism defined in paper [TA]. Note that we must have $x = q_1 \dots q_l = (p_1 \dots p_k)^{-1}$.

Commutativity isomorphism: The commutativity isomorphisms

$$\begin{aligned} & \sigma: \langle X, A, B \rangle \rightarrow \langle X, {}^p B, A \rangle \\ & X \in \mathcal{C}_r, \quad A \in \mathcal{C}_p, \quad B \in \mathcal{C}_q, \quad rpq = 1 \end{aligned}$$

is defined as follows. Recall the braiding homeomorphism (see [TA]):

$$\tilde{b}: S_3(r, p, q; 1, 1, 1) \rightarrow S_3(r, pqp^{-1}, p; 1, p^{-1}, 1)$$

Then we define σ as the following composition:

$$\begin{aligned} & \tau[S_3(r, p, q; 1, 1, 1; X, A, B)] \xrightarrow{\tilde{b}_*} \tau[S_3(r, pqp^{-1}, p; 1, p^{-1}, 1; X, B, A)] \\ & \xrightarrow{T_{1,p,1}} \tau[S_3(r, pqp^{-1}, p; 1, 1, 1; X, {}^p B, A)] \end{aligned}$$

This completes the construction of MS data from a MF.

Now we need to check that so defined isomorphisms satisfy all the axioms of G -equivariant Moore-Seiberg data.

- Non-degeneracy of MS data follows from the non-degeneracy of modular functor.
- Normalization axiom of MS data follows from the normalization axiom of modular functor.
- Associativity of \mathcal{G} of MS data follows from the associativity of cuts of the complex $M(\tilde{\Sigma}, \Sigma)$
- Rotation axiom of MS data follows from the rotation axiom of the complex $M(\tilde{\Sigma}, \Sigma)$.
- Symmetry of \mathcal{G} in MS data follows from the symmetry of \mathbf{F} move in the complex $M(\tilde{\Sigma}, \Sigma)$.
- ϕ -relation and the compatibility of ϕ follows from the \mathbf{P}_x relation of the complex $M(\tilde{\Sigma}, \Sigma)$ and the associativity of the group multiplication in G .
- Hexagon axiom of MS data follows from the braiding axiom of the complex $M(\tilde{\Sigma}, \Sigma)$.
- Dehn twist axiom of MS data follows from the Dehn twist axiom of the complex $M(\tilde{\Sigma}, \Sigma)$.

Most of the above correspondences are easy to check since they directly follow from the definition. For illustration, we will demonstrate the Dehn twist axiom for MS data.

If we look at the Dehn twist axiom of MS data and rewrite everything using our definition of conformal block, replacing $\langle A, B \rangle$ by $\tau[S_2(p, q; 1, 1; [A, B])]$ etc, we get the following diagram (as before, $A \in \mathcal{C}_p, B \in \mathcal{C}_q, pq = 1$).

$$\begin{array}{ccccc}
 \tau[S_2(p, q; 1, 1; [A, B])] & \xrightarrow{\sigma} & \tau[S_2(q, p; 1, 1; [{}^p B, A])] & & \\
 \downarrow Z & & \searrow Z & & \\
 \tau[S_2(q, p; 1, 1; [B, A])] & \xrightarrow{\sigma} & \tau[S_2(p, q; 1, 1; [{}^q A, B])] & \xrightarrow{\varphi} & \tau[S_2(p, q; 1, 1; [A, {}^p B])]
 \end{array}$$

FIGURE 7. Rewriting Dehn twist axiom

Now it is easy to derive the Dehn twist axiom of MS data from the axioms of modular functor. Recall that in the complex $M(\tilde{\Sigma}, \Sigma)$, we had the following relation, which we also called “Dehn twist axiom”: (we assume α is the boundary associated with A and the label $(p, 1)$ and β is the boundary circle associated with B and the label $(q, 1)$; as before, $pq = 1$)

$$\begin{array}{ccccc}
 S_2(p, q; 1, 1) & \xrightarrow{B_{\alpha, \beta}} & S_2(q, p; q, 1) & \xrightarrow{Z} & \\
 \downarrow Z & & \searrow \varphi_p & & \\
 S_2(q, p; 1, 1) & \xrightarrow{B_{\beta, \alpha}} & S_2(p, q; p, 1) & & S_2(p, q; 1, q)
 \end{array}$$

The above relation is a certian relation between homeomorphism of G -covers; bu functoriality axiom of modular functor, this relation must also hold between the

corresponding modular functor spaces, which exactly gives us relation of Figure 7, this proving the Dehn twist axiom for MS data.

The proof of all other axioms of MS data is done in a similar manner, by rewriting the axioms in terms of modular functor spaces for standard blocks and using relations in the complex $M(\tilde{\Sigma}, \Sigma)$.

This finishes the proof of our main theorem.

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